ABOUT CURVATURE, CONFORMAL METRICS AND WARPED PRODUCTS

FERNANDO DOBARRO
&
BÜLENT ÜNAL

Abstract. We consider the curvature of a family of warped products of two pseduo-Riemannian manifolds (B, g_B) and (F, g_F) furnished with metrics of the form $c^2 g_B \oplus w^2 g_F$ and, in particular, of the type $w^{2\mu} g_B \oplus w^2 g_F$, where $c, w \colon B \to (0, \infty)$ are smooth functions and μ is a real parameter. We obtain suitable expressions for the Ricci tensor and scalar curvature of such products that allow us to establish results about the existence of Einstein or constant scalar curvature structures in these categories. If (B, g_B) is Riemannian, the latter question involves nonlinear elliptic partial differential equations with concave-convex nonlinearities and singular partial differential equations of the Lichnerowicz-York type among others.

1. Introduction

The main concern of this paper is the curvature of a special family of warped pseudo-metrics on product manifolds. We introduce a suitable form for the relations among the involved curvatures in such metrics and apply them to the existence and/or construction of Einstein and constant scalar curvature metrics in this family.

Let $B = (B_m, g_B)$ and $F = (F_k, g_F)$ be two pseudo-Riemannian manifolds of dimensions $m \geq 1$ and $k \geq 0$, respectively and also let $B \times F$ be the usual product manifold of B and F. For a given smooth function $w \in C^\infty_{>0}(B) = \{v \in C^\infty(B) : v(x) > 0, \forall\, x \in B\}$, the *warped product* $B \times_w F = ((B \times_w F)_{m+k}, g = g_B + w^2 g_F)$ was defined by Bishop and O'Neill in [19] in order to study manifolds of negative curvature.

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In this article, we deal with a particular class of warped products, i.e. when the pseudo-metric in the base is affected by a conformal change. Precisely, for given smooth functions $c, w \in C^{\infty}_{>0}(B)$ we will call $((B \times F)_{m+k}, g = c^2 g_B + w^2 g_F)$ as a $[c,w]$-base conformal warped product (briefly $[c,w]$-*bcwp*), denoted by $B \times_{[c,w]} F$. We will concentrate our attention on a special subclass of this structure, namely when there is a relation between the *conformal factor* c and the *warping function* w of the form $c = w^{\mu}$, where μ is a real parameter and we will call the $[\psi^{\mu}, \psi]$-*bcwp* as a (ψ, μ)-*bcwp*. Note that we generically called the latter case as special base conformal warped products, briefly *sbcwp* in [29].

As we will explain in §2, metrics of this type play a relevant role in several topics of differential geometry and theoretical physics (see also [29]). This article concerns curvature related questions of these metrics which are of interest not only in the applications, but also from the points of view of differential geometry and the type of the involved nonlinear partial differential equations (PDE), such as those with concave-convex nonlinearities and the Lichnerowicz-York equations.

The article is organized in the following way: in §2 after a brief description of several fields where pseudo-metrics described as above are applied, we formulate the curvature problems that we deal within the next sections and give the statements of the main results. In §3, we state Theorems 2.2 and 2.3 in order to express the Ricci tensor and scalar curvature of a (ψ, μ)-*bcwp* and sketch their proofs (see [29, Section 3] for detailed computations). In §4 and 5, we establish our main results about the existence of (ψ, μ)-*bcwp*'s of constant scalar curvature with compact Riemannian base.

2. Motivations and Main results

As we announced in the introduction, we firstly want to mention some of the major fields of differential geometry and theoretical physics where base conformal warped products are applied.

i: In the construction of a large class of non trivial static anti de Sitter vacuum space-times
- In the Schwarzschild solutions of the Einstein equations (see [10, 18, 41, 59, 69, 74]).
- In the Riemannian Schwarzschild metric, namely (see [10]).
- In the *"generalized Riemannian anti de Sitter* $\mathbf{T}^2$ *black hole metrics"* (see §3.2 of [10] for details).
- In the Bañados-Teitelboim-Zanelli (BTZ) and de Sitter (dS) black holes (see [1, 15, 16, 28, 45] for details).

Indeed, all of them can be generated by an approach of the following type: let (F_2, g_F) be a pseudo-Riemannian manifold and g be

a pseudo-metric on $\mathbb{R}_+ \times \mathbb{R} \times F_2$ defined by

$$g = \frac{1}{u^2(r)} dr^2 \pm u^2(r) dt^2 + r^2 g_F. \tag{2.1}$$

After the change of variables $s = r^2$, $y = \frac{1}{2}t$, there results $ds^2 = 4r^2 dr^2$ and $dy^2 = \frac{1}{4}dt^2$. Then (2.1) is equivalent to

$$\begin{aligned} g &= \frac{1}{\sqrt{s}}\Big[\frac{1}{4\sqrt{s}u^2(\sqrt{s})} ds^2 \pm 4\sqrt{s}u^2(\sqrt{s})dy^2\Big] + s g_F \\ &= (s^{\frac{1}{2}})^{2(-\frac{1}{2})}\Big[(2s^{\frac{1}{4}}u(s^{\frac{1}{2}}))^{2(-1)}ds^2 \pm (2s^{\frac{1}{4}}u(s^{\frac{1}{2}}))^2 dy^2\Big] + (s^{\frac{1}{2}})^2 g_F. \end{aligned} \tag{2.2}$$

Note that roughly speaking, g is a nested application of two (ψ, μ)-*bcwp*'s. That is, on $\mathbb{R}_+ \times \mathbb{R}$ and taking

$$\psi_1(s) = 2s^{\frac{1}{4}}u(s^{\frac{1}{2}}) \text{ and } \mu_1 = -1, \tag{2.3}$$

the metric inside the brackets in the last member of (2.2) is a (ψ_1, μ_1)-*bcwp*, while the metric g on $(\mathbb{R}_+ \times \mathbb{R}) \times F_2$ is a (ψ_2, μ_2)-*bcwp* with

$$\psi_2(s, y) = s^{\frac{1}{2}} \text{ and } \mu_2 = -\frac{1}{2}. \tag{2.4}$$

In the last section of [29], through the application of Theorems 2.2 and 2.3 below and several standard computations, we generalized the latter approach to the case of an Einstein fiber (F_k, g_F) with dimension $k \geq 2$.

ii: In the study of the equivariant isometric embeddings of space-time slices in Minkowski spaces (see [39, 38]).

iii: In the Kaluza-Klein theory (see [76, §7.6, Particle Physics and Geometry], [60] and [77]) and in the Randall-Sundrum theory [30, 40, 63, 64, 65, 71] with μ as a free parameter. For example, in [46] the following metric is considered

$$e^{2\mathcal{A}(y)} g_{ij} dx^i dx^j + e^{2\mathcal{B}(y)} dy^2, \tag{2.5}$$

with the notation $\{x^i\}$, $i = 0, 1, 2, 3$ for the coordinates in the 4-dimensional space-time and $x^5 = y$ for the fifth coordinate on an extra dimension. In particular, Ito takes the ansatz

$$\mathcal{B} = \alpha\mathcal{A}, \tag{2.6}$$

which corresponds exactly to our *sbcwp* metrics, considering $g_B = dy^2$, $g_F = g_{ij} dx^i dx^j$, $\psi(y) = e^{\frac{\mathcal{B}(y)}{\alpha}} = e^{\mathcal{A}(y)}$ and $\mu = \alpha$.

iii: In String and Supergravity theories, for instance, in the Maldacena conjecture about the duality between compactifications of M/string

theory on various Anti-de Sitter space-times and various conformal field theories (see [55, 62]) and in warped compactifications (see [40, 72] and references therein). Besides all of these, there are also frequent occurrences of this type of metrics in string topics (see [33, 34, 35, 36, 37, 53, 61, 71] and also [1, 12, 67] for some reviews about these topics).

iv: In the derivation of effective theories for warped compactification of supergravity and the Hořava-Witten model (see [50, 51]). For instance, in [51] the ansatz $ds^2 = h^\alpha ds^2(X_4) + h^\beta ds^2(Y)$ is considered where X_4 is a four-dimensional space-time with coordinates x^μ, Y is a Calabi-Yau manifold (the so called internal space) and h depends on the four-dimensional coordinates x^μ, in order to study the dynamics of the four-dimensional effective theory. We note that in those articles, the structure of the expressions of the Ricci tensor and scalar curvature of the involved metrics result particularly useful. We observe that they correspond to very particular cases of the expressions obtained by us in [29], see also Theorems 2.2 and 2.3 and Proposition 2.4 stated below.

v: In the discussion of Birkhoff-type theorems (generally speaking these are the theorems in which the gravitational vacuum solutions admit more symmetry than the inserted metric ansatz, (see [41, page 372] and [17, Chapter 3]) for rigorous statements), especially in Equation 6.1 of [66] where, H-J. Schmidt considers a special form of a *bcwp* and basically shows that if a *bcwp* of this form is Einstein, then it admits one Killing vector more than the fiber. In order to achieve that, the author considers for a specific value of μ, namely $\mu = (1-k)/2$, in the following problem:

> *Does there exist a smooth function $\psi \in C^\infty_{>0}(B)$ such that the corresponding (ψ, μ)*-bcwp $(B_2 \times F_k, \psi^{2\mu} g_B + \psi^2 g_F)$ *is an Einstein manifold?* (see also **(Pb-Eins.)** below.)

vi: In the study of bi-conformal transformations, bi-conformal vector fields and their applications (see [32, Remark in Section 7] and [31, Sections 7 and 8]).

vii: In the study of the spectrum of the Laplace-Beltrami operator for $p-$forms. For instance in Equation (1.1) of [11], the author considers the structure that follows: let $\overline{M}$ be an n-dimensional compact, Riemannian manifold with boundary, and let y be a boundary-defining function; she endows the interior M of $\overline{M}$ with a Riemannian metric ds^2 such that in a small tubular neighborhood of ∂M in M, ds^2 takes the form

$$ds^2 = e^{-2(a+1)t} dt^2 + e^{-2bt} d\theta^2_{\partial M}, \tag{2.7}$$

where $t := -\log y \in (c, +\infty)$ and $d\theta^2_{\partial M}$ is the Riemannian metric on ∂M (see [11, 56] and references therein for details).

Notation 2.1. From now on, we will use the Einstein summation convention over repeated indices and consider only connected manifolds. Furthermore, we will denote the Laplace-Beltrami operator on a pseudo-Riemannian manifold (N, h) by $\Delta_N(\cdot)$, i.e., $\Delta_N(\cdot) = \nabla^{N\,i}\nabla^{N}{}_i(\cdot)$. Note that Δ_N is elliptic if (N, h) is Riemannian and it is hyperbolic when (N, h) is Lorentzian. If (N, h) is neither Riemannian nor Lorentzian, then the operator is ultra-hyperbolic.

Furthermore, we will consider the Hessian of a function $v \in C^\infty(N)$, denoted by H^v_h or H^v_N, so that the second covariant differential of v is given by $H^v_h = \nabla(\nabla v)$. Recall that the Hessian is a symmetric $(0, 2)$ tensor field satisfying

$$H^v_h(X, Y) = XYv - (\nabla_X Y)v = h(\nabla_X(\text{grad } v), Y), \tag{2.8}$$

for any smooth vector fields X, Y on N.

For a given pseudo-Riemannian manifold $N = (N, h)$ we will denote its Riemann curvature tensor, Ricci tensor and scalar curvature by R_N, Ric_N and S_N, respectively.

We will denote the set of all lifts of all vector fields of B by $\mathfrak{L}(B)$. Note that the lift of a vector field X on B denoted by $\widetilde{X}$ is the vector field on $B \times F$ given by $\mathrm{d}\pi(\widetilde{X}) = X$ where $\pi\colon B \times F \to B$ is the usual projection map.

In Section 3, we will sketch the proofs of the following two theorems related to the Ricci tensor and the scalar curvature of a generic (ψ, μ)-*bcwp*.

Theorem 2.2. *Let $B = (B_m, g_B)$ and $F = (F_k, g_F)$ be two pseudo-Riemannian manifolds with $m \geq 3$ and $k \geq 1$, respectively and also let $\mu \in \mathbb{R} \setminus \{0, 1, \overline{\mu}, \overline{\mu}_\pm\}$ be a real number with*

$$\overline{\mu} := -\frac{k}{m-2} \text{ and } \overline{\mu}_\pm := \overline{\mu} \pm \sqrt{\overline{\mu}^2 - \overline{\mu}}.$$

Suppose $\psi \in C^\infty_{>0}(B)$. Then the Ricci curvature tensor of the corresponding (ψ, μ)-bcwp, denoted by Ric verifies the relation

$$\begin{aligned}
&Ric = Ric_B + \beta^H \frac{1}{\psi^{\frac{1}{\alpha^H}}} \mathrm{H}_B^{\psi^{\frac{1}{\alpha^H}}} - \beta^\Delta \frac{1}{\psi^{\frac{1}{\alpha^\Delta}}} \Delta_B \psi^{\frac{1}{\alpha^\Delta}} g_B \text{ on } \mathcal{L}(B) \times \mathcal{L}(B),\\
&Ric = 0 \text{ on } \mathcal{L}(B) \times \mathcal{L}(F),\\
&Ric = Ric_F - \frac{1}{\psi^{2(\mu-1)}} \frac{\beta^\Delta}{\mu} \frac{1}{\psi^{\frac{1}{\alpha^\Delta}}} \Delta_B \psi^{\frac{1}{\alpha^\Delta}} g_F \text{ on } \mathcal{L}(F) \times \mathcal{L}(F),
\end{aligned} \tag{2.9}$$

where

$$
\begin{aligned}
\alpha^{\Delta} &= \frac{1}{(m-2)\mu+k}, \\
\beta^{\Delta} &= \frac{\mu}{(m-2)\mu+k}, \\
\alpha^{H} &= \frac{-[(m-2)\mu+k]}{\mu[(m-2)\mu+k]+k(\mu-1)}, \\
\beta^{H} &= \frac{[(m-2)\mu+k]^2}{\mu[(m-2)\mu+k]+k(\mu-1)}.
\end{aligned} \tag{2.10}
$$

Theorem 2.3. *Let $B=(B_m,g_B)$ and $F=(F_k,g_F)$ be two pseudo-Riemannian manifolds of dimensions $m\geq 2$ and $k\geq 0$, respectively. Suppose that S_B and S_F denote the scalar curvatures of $B=(B_m,g_B)$ and $F=(F_k,g_F)$, respectively. If $\mu\in\mathbb{R}$ and $\psi\in C^{\infty}_{>0}(B)$, then the scalar curvature S of the corresponding (ψ,μ)-bcwp verifies,*

(i) *If $\mu\neq -\frac{k}{m-1}$, then*

$$-\beta\Delta_B u+S_B u=Su^{2\mu\alpha+1}-S_F u^{2(\mu-1)\alpha+1} \tag{2.11}$$

where

$$\alpha=\frac{2[k+(m-1)\mu]}{\{[k+(m-1)\mu]+(1-\mu)\}k+(m-2)\mu[k+(m-1)\mu]}, \tag{2.12}$$

$$\beta=\alpha 2[k+(m-1)\mu]>0 \tag{2.13}$$

and $\psi=u^{\alpha}>0$.

(ii) *If $\mu=-\frac{k}{m-1}$, then*

$$-k\left[1+\frac{k}{m-1}\right]\frac{|\nabla^B\psi|_B^2}{\psi^2}=\psi^{-2\frac{k}{m-1}}[S-S_F\psi^{-2}]-S_B. \tag{2.14}$$

From the mathematical and physical points of view, there are several interesting questions about (ψ,μ)-*bcwp*'s. In [29] we began the study of existence and/or construction of Einstein (ψ,μ)-*bcwp*'s and those of constant scalar curvature. These questions are closely connected to Theorems 2.2 and 2.3.

In [29], by applying Theorem 2.2, we give suitable conditions that allow us to study some particular cases of the problem:

> **(Pb-Eins.)** Given $\mu\in\mathbb{R}$, does there exist a smooth function $\psi\in C^{\infty}_{>0}(B)$ such that the corresponding (ψ,μ)-*bcwp* is an Einstein manifold?

In particular, we obtain the following result as an immediate corollary of Theorem 2.2.

Proposition 2.4. *Let us assume the hypothesis of Theorem 2.2. Then the corresponding (ψ,μ)-bcwp is an Einstein manifold with λ if and only if (F,g_F) is Einstein with ν constant and the system that follows is verified*

(2.15)

$$\lambda\psi^{2\mu}g_B = \mathrm{Ric}_B + \beta^H \frac{1}{\psi^{\frac{1}{\alpha^H}}}\mathrm{H}_B^{\psi^{\frac{1}{\alpha^H}}} - \beta^\Delta \frac{1}{\psi^{\frac{1}{\alpha^\Delta}}}\Delta_B \psi^{\frac{1}{\alpha^\Delta}} g_B \ \text{on } \mathcal{L}(B)\times\mathcal{L}(B)$$

$$\lambda\psi^2 = \nu - \frac{1}{\psi^{2(\mu-1)}}\frac{\beta^\Delta}{\mu}\frac{1}{\psi^{\frac{1}{\alpha^\Delta}}}\Delta_B\psi^{\frac{1}{\alpha^\Delta}},$$

where the coefficients are given by (2.10).

Compare the system (2.15) with the well known one for a classical warped product in [18, 49, 59]. By studying (2.15), we have obtained the generalization of the construction exposed in the above motivational examples in **i** and **v**, among other related results. We suggest the interested reader consider the results about the problem **(Pb-Eins.)** stated in [29].

Now, we focus on the problems which we will deal in §4. Let $B=(B_m,g_B)$ and $F=(F_k,g_F)$ be pseudo-Riemannian manifolds.
There is an extensive number of publications about the well known Yamabe problem namely:

> **(Ya)** [79, 75, 68, 13] Does there exist a function $\varphi\in C^\infty_{>0}(B)$ such that $(B_m,\varphi^{\frac{4}{m-2}}g_B)$ has constant scalar curvature?

Analogously, in several articles the following problem has been studied:

> **(cscwp)** [27] Is there a function $w\in C^\infty_{>0}(B)$ such that the warped product $B\times_w F$ has constant scalar curvature?

In the sequel we will suppose that $B=(B_m,g_B)$ is a Riemannian manifold.

Thus, both problems bring to the study of the existence of positive solutions for nonlinear elliptic equations on Riemannian manifolds. The involved nonlinearities are powers with Sobolev critical exponent for the Yamabe problem and sub-linear (linear if the dimension k of the fiber is 3) for the problem of constant scalar curvature of a warped product.

In Section 4, we deal with a mixed problem between **(Ya)** and **(cscwp)** which is already proposed in [29], namely:

> **(Pb-sc)** Given $\mu \in \mathbb{R}$, does there exist a function $\psi \in C^{\infty}_{>0}(B)$ such that the corresponding (ψ, μ)-*bcwp* has constant scalar curvature?

*Note that when $\mu = 0$, **(Pb-sc)** corresponds to the problem **(cscwp)**, whereas when the dimension of the fiber $k = 0$ and $\mu = 1$, then **(Pb-sc)** corresponds to **(Ya)** for the base manifold. Finally **(Pb-sc)** corresponds to **(Ya)** for the usual product metric with a conformal factor in $C^{\infty}_{>0}(B)$ when $\mu = 1$.*

Under the hypothesis of **Theorem 2.3 i**, the analysis of the problem **(Pb-sc)** brings to the study of the existence and multiplicity of solutions $u \in C^{\infty}_{>0}(B)$ of

$$-\beta\Delta_B u + S_B u = \lambda u^{2\mu\alpha+1} - S_F u^{2(\mu-1)\alpha+1}, \tag{2.16}$$

where all the components of the equation are like in **Theorem 2.3 i** and λ (the conjectured constant scalar curvature of the corresponding (ψ, μ)-*bcwp*) is a real parameter. We observe that an easy argument of separation of variables, like in [24, §2] and [27], shows that there exists a positive solution of (2.16) only if the scalar curvature of the fiber S_H is constant. Thus this will be a natural assumption in the study of **(Pb-sc)**.
Furthermore, note that the involved nonlinearities in the right hand side of (2.16) dramatically change with the choice of the parameters, an exhaustive analysis of these changes is the subject matter of [29, §6].
There are several partial results about semi-linear elliptic equations like (2.16) with different boundary conditions, see for instance [2, 5, 6, 9, 21, 23, 26, 73, 78] and references in [29].

*In this article we will state our first results about the problem **(Pb-sc)** when the base B is a compact Riemannian manifold of dimension $m \geq 3$ and the fiber F has non-positive constant scalar curvature S_F.*

For brevity of our study, it will be useful to introduce the following notation: $\mu_{sc} := \mu_{sc}(m,k) = -\dfrac{k}{m-1}$ and $\mu_{p_Y} = \mu_{p_Y}(m,k) := -\dfrac{k+1}{m-2}$ (*sc* as scalar curvature and Y as Yamabe). Notice that $\mu_{p_Y} < \mu_{sc} < 0$.

We plan to study the case of $\mu = \mu_{sc}$ in a preceding project, therefore the related results are not going to be presented here.

We can synthesize our results about **(Pb-sc)** in the case of non-positive S_F as follow.

- *The case of scalar flat fiber, i.e. $S_F = 0$.*

  Theorem 2.5. *If $\mu \in (\mu_{p_Y}, \mu_{sc}) \cup (\mu_{sc}, +\infty)$ the answer to **(Pb-sc)** is affirmative.*

By assuming some additional restrictions on the scalar curvature of the base S_B, we obtain existence results for the range $\mu \leq \mu_{p_Y}$.

- *The case of fiber with negative constant scalar curvature, i.e.* $S_F < 0$. In order to describe the $\mu-$ranges of validity of the results, we will apply the notations introduced in [29, §5] (see Appendix A for a brief introduction of these notations).

  Theorem 2.6. *If "$(m, k) \in D$ and $\mu \in (0, 1)$" or "$(m, k) \in \mathcal{C}D$ and $\mu \in (0, 1) \cap (\mu_-, \mu_+)$" or "$(m, k) \in \mathcal{C}D$ and $\mu \in (0, 1) \cap \mathcal{C}[\mu_-, \mu_+]$", then the answer to **(Pb-sc)** is affirmative.*

  Remark 2.7. The first two cases in Theorem 2.6 will be studied by adapting the ideas in [5] and the last case by applying the results in [73, p. 99]. In the former - Theorem 4.15, the involved nonlinearities correspond to the so called concave-convex whereas in the latter - Theorem 4.16, they are singular as in the Lichnerowicz-York equation about the constraints for the Einstein equations (see [22], [43], [58], [57, p. 542-543] and [73, Chp.18]).
  Similarly to the case of $S_F = 0$, we obtain existence results for some remaining $\mu-$ranges by assuming some additional restrictions for the scalar curvature of the base S_B.

Naturally the study of **(Pb-sc)** allows us to obtain partial results of the related question:

> Given $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ does there exist a function $\psi \in C^\infty_{>0}(B)$ such that the corresponding (ψ, μ)-*bcwp* has constant scalar curvature λ?

These are stated in the several theorems and propositions in §4.

3. The curvature relations - Sketch of the proofs

The proofs of Theorems 2.2 and 2.3 require long and yet standard computations of the Riemann and Ricci tensors and the scalar curvature of a general base conformal warped product. Here, we reproduce the results for the Ricci tensor and the scalar curvature, and we also suggest the reader see [29, §3] for the complete computations.

Theorem 3.1. *The Ricci tensor of* $[c, w]$-bcwp*, denoted by Ric satisfies*

$$
\begin{aligned}
(1)\quad Ric = Ric_B - & \left[(m-2)\frac{1}{c}\mathrm{H}^c_B + k\frac{1}{w}\mathrm{H}^w_B\right] \\
& +2(m-2)\frac{1}{c^2}dc \otimes dc + k\frac{1}{wc}[dc \otimes dw + dw \otimes dc] \\
& - \left[(m-3)\frac{g_B(\nabla^B c, \nabla^B c)}{c^2} + \frac{\Delta_B c}{c} + k\frac{g_B(\nabla^B w, \nabla^B c)}{wc}\right] g_B \\
& \text{on } \mathcal{L}(B) \times \mathcal{L}(B),
\end{aligned}
$$

(2) $Ric = 0$ *on* $\mathcal{L}(B) \times \mathcal{L}(F)$,

(3) $$Ric = Ric_F - \frac{w^2}{c^2}\left[(m-2)\frac{g_B(\nabla^B w, \nabla^B c)}{wc} + \frac{\Delta_B w}{w} + (k-1)\frac{g_B(\nabla^B w, \nabla^B w)}{w^2}\right] g_F \text{ on } \mathcal{L}(F) \times \mathcal{L}(F).$$

Theorem 3.2. *The scalar curvature S of a* $[c,w]$-bcwp *is given by*

$$\begin{aligned} c^2 S &= S_B + S_F \frac{c^2}{w^2} - 2(m-1)\frac{\Delta_B c}{c} - 2k\frac{\Delta_B w}{w} \\ &- (m-4)(m-1)\frac{g_B(\nabla^B c, \nabla^B c)}{c^2} \\ &- 2k(m-2)\frac{g_B(\nabla^B w, \nabla^B c)}{wc} \\ &- k(k-1)\frac{g_B(\nabla^B w, \nabla^B w)}{w^2}. \end{aligned}$$

The following two lemmas (3.3 and 3.7) play a central role in the proof of Theorems 2.2 and 2.3. Indeed, it is sufficient to apply them in a suitable mode and make use of Theorems 3.1 and 3.2 several times, the reader can find all the details in [29, §2 and 4].

Let $N = (N_n, h)$ be a pseudo-Riemannian manifold of dimension n, $|\nabla(\cdot)|^2 = |\nabla^N(\cdot)|^2_N = h(\nabla^N(\cdot), \nabla^N(\cdot))$ and $\Delta_h = \Delta_N$.

Lemma 3.3. *Let L_h be a differential operator on $C^\infty_{>0}(N)$ defined by*

$$L_h v = \sum_{i=1}^{k} r_i \frac{\Delta_h v^{a_i}}{v^{a_i}}, \tag{3.1}$$

where $r_i, a_i \in \mathbb{R}$ and $\zeta := \sum_{i=1}^{k} r_i a_i$, $\eta := \sum_{i=1}^{k} r_i a_i^2$. Then,

(i)

$$L_h v = (\eta - \zeta)\frac{\|grad\ _h v\|^2_h}{v^2} + \zeta\frac{\Delta_h v}{v}. \tag{3.2}$$

(ii) *If $\zeta \neq 0$ and $\eta \neq 0$, for $\alpha = \dfrac{\zeta}{\eta}$ and $\beta = \dfrac{\zeta^2}{\eta}$, then we have*

$$L_h v = \beta \frac{\Delta_h v^{\frac{1}{\alpha}}}{v^{\frac{1}{\alpha}}}. \tag{3.3}$$

Remark 3.4. We also applied the latter lemma in the study of curvature of multiply warped products (see [28]).

Corollary 3.5. *Let L_h be a differential operator defined by*

$$L_h v = r_1 \frac{\Delta_h v^{a_1}}{v^{a_1}} + r_2 \frac{\Delta_h v^{a_2}}{v^{a_2}} \text{ for } v \in C^\infty_{>0}(N), \tag{3.4}$$

where $r_1 a_1 + r_2 a_2 \neq 0$ and $r_1 a_1^2 + r_2 a_2^2 \neq 0$. Then, by changing the variables $v = u^\alpha$ with $0 < u \in C^\infty(N)$, $\alpha = \dfrac{r_1 a_1 + r_2 a_2}{r_1 a_1^2 + r_2 a_2^2}$ and $\beta = \dfrac{(r_1 a_1 + r_2 a_2)^2}{r_1 a_1^2 + r_2 a_2^2} = \alpha(r_1 a_1 + r_2 a_2)$ there results

$$L_h v = \beta \frac{\Delta_h u}{u}. \tag{3.5}$$

Remark 3.6. By the change of variables as in Corollary 3.5 equations of the type

$$L_h v = r_1 \frac{\Delta_h v^{a_1}}{v^{a_1}} + r_2 \frac{\Delta_h v^{a_2}}{v^{a_2}} = H(v, x, s), \tag{3.6}$$

transform into

$$\beta \Delta_h u = u H(u^\alpha, x, s). \tag{3.7}$$

Lemma 3.7. *Let $\mathcal{H}_h$ be a differential operator on $C^\infty_{>0}(N)$ defined by*

$$\mathcal{H}_h v = \sum r_i \frac{H_h^{v^{a_i}}}{v^{a_i}}, \tag{3.8}$$

$\zeta := \sum r_i a_i$ and $\eta := \sum r_i a_i^2$, where the indices extend from 1 to $l \in \mathbb{N}$ and any $r_i, a_i \in \mathbb{R}$. Hence,

$$\mathcal{H}_h v = (\eta - \zeta) \frac{1}{v^2} dv \otimes dv + \zeta \frac{1}{v} H_h^v, \tag{3.9}$$

where $\otimes$ is the usual tensorial product. If furthermore, $\zeta \neq 0$ and $\eta \neq 0$, then

$$\mathcal{H}_h v = \beta \frac{H_h^{v^{\frac{1}{\alpha}}}}{v^{\frac{1}{\alpha}}}, \tag{3.10}$$

where $\alpha = \dfrac{\zeta}{\eta}$ and $\beta = \dfrac{\zeta^2}{\eta}$.

4. The problem **(Pb-sc)** - Existence of solutions

Throughout this section, we will assume that B is not only a Riemannian manifold of dimension $m \geq 3$, but also "compact" and connected. We further assume that F is a pseudo-Riemannian manifold of dimension $k \geq 0$ with constant scalar curvature $S_F \leq 0$. Moreover, we will assume that $\mu \neq \mu_{sc}$.

Hence, we will concentrate our attention on the relations (2.11), (2.12) and (2.13) by applying Theorem 2.3 (i).

Let λ_1 denote the principal eigenvalue of the operator

$$L(\cdot) = -\beta\Delta_B(\cdot) + S_B(\cdot), \tag{4.1}$$

and $u_1 \in C^\infty_{>0}(B)$ be the corresponding positive eigenfunction with $\|u_1\|_\infty = 1$, where β is as in Theorem 2.3.

First of all, we will state some results about uniqueness and non-existence of positive solutions for Equation (2.16) under the latter hypothesis.

About the former, we adapt Lemma 3.3 in [5, p. 525] to our situation (for a detailed proof see [5], [20, Method II, p. 103] and also [70]).

Lemma 4.1. *Let $f \in C^0(\mathbb{R}_{>0})$ such that $t^{-1}f(t)$ is decreasing. If v and w satisfy*

$$\begin{gathered} -\beta\Delta_B v + S_B v \leq f(v), \\ v \in C^\infty_{>0}(B), \end{gathered} \tag{4.2}$$

and

$$\begin{gathered} -\beta\Delta_B w + S_B w \geq f(w), \\ w \in C^\infty_{>0}(B), \end{gathered} \tag{4.3}$$

then $w \geq v$ on B.

Proof. Let $\theta(t)$ be a smooth nondecreasing function such that $\theta(t) \equiv 0$ for $t \leq 0$ and $\theta(t) \equiv 1$ for $t \geq 1$. Thus for all $\epsilon > 0$,

$$\theta_\epsilon(t) := \theta\left(\frac{t}{\epsilon}\right)$$

is smooth, nondecreasing, nonnegative and $\theta(t) \equiv 0$ for $t \leq 0$ and $\theta(t) \equiv 1$ for $t \geq \epsilon$. Furthermore $\gamma_\epsilon(t) := \int_0^t s\theta'_\epsilon(s)ds$ satisfies $0 \leq \gamma_\epsilon(t) \leq \epsilon$, for any $t \in \mathbb{R}$.
On the other hand, since (B, g_B) is a compact Riemannian manifold without boundary and $\beta > 0$, like in [5, Lemma 3.3, p. 526] there results

$$\int_B [-v\beta\Delta_B w + w\beta\Delta_B v]\theta_\epsilon(v-w)dv_{g_B} \leq \int_B [-\beta\Delta_B v]\gamma_\epsilon(v-w)dv_{g_B}. \tag{4.4}$$

Hence, by the above considerations about θ_ϵ and γ_ϵ, (4.4) implies that

$$\int_B [-v\beta\Delta_B w + w\beta\Delta_B v]\theta_\epsilon(v-w)dv_{g_B} \leq \epsilon \int_{[-\beta\Delta_B v \geq 0]} [-\beta\Delta_B v]dv_{g_B}. \tag{4.5}$$

Now, by applying (4.2) and (4.3) there results
(4.6)

$$-v\beta\Delta_B w + w\beta\Delta_B v = vLw - wLv \geq vf(w) - wf(v) = vw\left[\frac{f(w)}{w} - \frac{f(v)}{v}\right].$$

Thus by combining (4.6) and (4.5), as $\epsilon \to 0^+$ we led to

$$\int_{[v>w]} vw \left[\frac{f(w)}{w} - \frac{f(v)}{v}\right] dv_{g_B} \leq 0 \tag{4.7}$$

and conclude the proof like in [5, Lemma 3.3, p. 526-527]. But $\dfrac{f(v)}{v} < \dfrac{f(w)}{w}$ on $[v > w]$ and hence $\mathrm{meas}[v > w] = 0$; thus $v \leq w$. [1] □

Corollary 4.2. *Let $f \in C^0(\mathbb{R}_{>0})$ such that $t^{-1}f(t)$ is decreasing. Then*

$$\begin{gathered} -\beta\Delta_B v + S_B v = f(v), \\ v \in C^\infty_{>0}(B) \end{gathered} \tag{4.8}$$

has at most one solution.

Proof. Assume that v and w are two solutions of (4.8). Then by applying Lemma 4.1 firstly with v and w, and conversely with w and v, the conclusion is proved. □

Remark 4.3. Notice that Lemma 4.1 and Corollary 4.2 allow the function $f \in C^0(\mathbb{R}_{>0})$ to be singular at 0.

Related to the non-existence of smooth positive solutions for Equation (2.16), we will state an easy result under the general hypothesis of this section.

Proposition 4.4. *If either $\max_B S_B \leq \inf_{u\in\mathbb{R}_{>0}} u^{2\mu\alpha}(\lambda - S_F u^{-2\alpha})$ or $\min_B S_B \geq \sup_{u\in\mathbb{R}_{>0}} u^{2\mu\alpha}(\lambda - S_F u^{-2\alpha})$, then (2.16) has no solution in $C^\infty_{>0}(B)$.*

Proof. It is sufficient to apply the maximum principle with some easy adjustments to the particular involved coefficients. □

- *The case of scalar flat fiber, i.e. $S_F = 0$.*

In this case, the term containing the nonlinearity $u^{2(\mu-1)\alpha+1}$ becomes noninfluent in (2.16), thus **(Pb-sc)** equivalently results to the study of existence of solutions for the problem:

$$\begin{gathered} -\beta\Delta_B u + S_B u = \lambda u^{2\mu\alpha+1}, \\ u \in C^\infty_{>0}(B), \end{gathered} \tag{4.9}$$

where λ is a real parameter (i.e., it is the searched constant scalar curvature) and $\psi = u^\alpha$.

[1] meas denotes the usual g_B−measure on the compact Riemannian manifold (B_m, g_B)

Remark 4.5. [2] Let $p \in \mathbb{R}\backslash\{1\}$ and $(\lambda_0, u_0) \in (\mathbb{R}\backslash\{0\}) \times C^\infty_{>0}(B)$ be a solution of

$$\begin{aligned} -\beta\Delta_B u + S_B u = \lambda u^p, \\ u \in C^\infty_{>0}(B). \end{aligned} \tag{4.10}$$

Hence, by the difference of homogeneity between both members of (4.9), it is easy to show that if $\lambda \in \mathbb{R}$ satisfies $\mathrm{sign}(\lambda) = \mathrm{sign}(\lambda_0)$, then (λ, u_λ) is a solution of (4.10), where $u_\lambda = t_\lambda u_0$ and $t_\lambda = \left(\dfrac{\lambda}{\lambda_0}\right)^{\frac{1}{1-p}}$.

Thus by (4.9), we obtain geometrically: if the parameter μ is given in a way that $p := 2\mu\alpha + 1 \neq 1$ and $B \times_{[\psi_0^\mu, \psi_0]} F$ has constant scalar curvature $\lambda_0 \neq 0$, then for any $\lambda \in \mathbb{R}$ verifying $\mathrm{sign}(\lambda) = \mathrm{sign}(\lambda_0)$, there results that $B \times_{[\psi_\lambda^\mu, \psi_\lambda]} F$ is of scalar curvature λ, where $\psi_\lambda = t_\lambda^\alpha \psi_0$ and t_λ given as above.

Theorem 4.6. ($Case$: $\mu = 0$) *The scalar curvature of a $(\psi, 0)$-bcwp of base B and fiber F (i.e., a singly warped product $B \times_\psi F$) is a constant λ if and only if $\lambda = \lambda_1$ and ψ is a positive multiple of $u_1^{\frac{2}{k+1}}$ (i.e., $\psi = tu_1^{\frac{2}{k+1}}$ for some $t \in \mathbb{R}_{>0}$).*

Proof. First of all note that $\mu = 0$ implies $\alpha = \dfrac{2}{k+1}$. On the other hand, in this case, the problem (4.9) is linear, so it is sufficient to apply the well known results about the principal eigenvalue and its associated eigenfunctions of operators like (4.1) in a suitable setting. □

Theorem 4.7. ($Case$: $\mu_{sc} < \mu < 0$) *The scalar curvature of a (ψ, μ)-bcwp of base B and fiber F is a constant λ, only if $sign(\lambda) = sign(\lambda_1)$. Furthermore,*

(1) *if $\lambda = 0$ then there exists $\psi \in C^\infty_{>0}(B)$ such that $B \times_{[\psi^\mu, \psi]} F$ has constant scalar curvature 0 if and only if $\lambda_1 = 0$. Moreover, such ψ's are the positive multiples of u_1^α, i.e. tu_1^α, $t \in \mathbb{R}_{>0}$.*
(2) *if $\lambda > 0$ then there exists $\psi \in C^\infty_{>0}(B)$ such that $B \times_{[\psi^\mu, \psi]} F$ has constant scalar curvature λ if and only if $\lambda_1 > 0$. In this case, the solution ψ is unique.*
(3) *if $\lambda < 0$ then there exists $\psi \in C^\infty_{>0}(B)$ such that $B \times_{[\psi^\mu, \psi]} F$ has constant scalar curvature λ when $\lambda_1 < 0$ and is close enough to 0.*

Proof. The condition $\mu_{sc} < \mu < 0$ implies that $0 < p := 2\mu\alpha + 1 < 1$, i.e., the problem (4.9) is sublinear. Thus, to prove the theorem one can use variational arguments as in [24] (alternatively, degree theoretic arguments as in [7] or bifurcation theory as in [27]).

[2]Along this article we consider the sign function defined by $\mathrm{sign} = \chi_{(0,+\infty)} - \chi_{(-\infty,0)}$, where χ_A is the characteristic function of the set A.

We observe that in order to obtain the positivity of the solutions required in (4.9), one may apply the maximum principle for the case of $\lambda > 0$ and the antimaximum principle for the case of $\lambda < 0$.
The uniqueness for $\lambda > 0$ is a consequence of Corollary 4.2. □

Remark 4.8. In order to consider the next case we introduce the following notation. For a given p such that $1 < p \leq p_Y$, let

$$\kappa_p := \inf_{v \in \mathcal{H}_p} \int_B \left(|\nabla^B v|^2 + \frac{S_B}{\beta} v^2 \right) dv_{g_B}, \tag{4.11}$$

where

$$\mathcal{H}_p := \left\{ v \in H^1(B) : \int_B |v|^{p+1} dv_{g_B} = 1 \right\}.$$

Now, we consider the following two cases.

$(1 < p < p_Y)$: In this case by adapting [42, Theorem 1.3], there exists $u_p \in C^\infty_{>0}(B)$ such that $(\beta\kappa_p, u_p)$ is a solution of (4.10) and $\displaystyle\int_B u_p^{p+1} dv_{g_B} = 1$.

$(p = p_Y)$: For this specific and important value, analogously to [42, §2], we distinguish three subcases along the study of our problem (4.10), in correspondence with the sign(κ_{p_Y}).

- $\underline{\kappa_{p_Y} = 0}$**:** in this case, there exists $u_{p_Y} \in C^\infty_{>0}(B)$ such that $(0, u_{p_Y})$ is a solution of (4.10) and $\displaystyle\int_B u_{p_Y}^{p_Y+1} dv_{g_B} = 1$.
- $\underline{\kappa_{p_Y} < 0}$**:** here there exists $u_{p_Y} \in C^\infty_{>0}(B)$ such that $(\beta\kappa_{p_Y}, u_{p_Y})$ is a solution of (4.10) and $\displaystyle\int_B u_{p_Y}^{p_Y+1} dv_{g_B} = 1$.
- $\underline{\kappa_{p_Y} > 0}$**:** this is a more difficult case, let K_m be the sharp Euclidean Sobolev constant

$$K_m = \sqrt{\frac{4}{m(m-2)\omega_m^{\frac{2}{m}}}}, \tag{4.12}$$

where ω_m is the volume of the unit $m-$sphere. Thus, if

$$\kappa_{p_Y} < \frac{1}{K_m^2}, \tag{4.13}$$

then there exists $u_{p_Y} \in C^\infty_{>0}(B)$ such that $(\beta\kappa_{p_Y}, u_{p_Y})$ is a solution of (4.10) and $\displaystyle\int_B u_{p_Y}^{p_Y+1} dv_{g_B} = 1$. Furthermore, the condition

$$\kappa_{p_Y} \leq \frac{1}{K_m^2} \tag{4.14}$$

is sharp by [42], so that this is independent of the underlying manifold and the potential considered.
The equality case in (4.14) is discussed in [44].

This results allow to establish the following two theorems.

Theorem 4.9. ($Cases$: $\mu_{p_Y} < \mu < \mu_{sc}$ *or* $0 < \mu$) *There exists* $\psi \in C^{\infty}_{>0}(B)$ *such that the scalar curvature of* $B \times_{[\psi^{\mu},\psi]} F$ *is a constant* λ *if and only if* $sign(\lambda) = sign(\kappa_p)$ *where* $p := 2\mu\alpha+1$ *and* κ_p *is given by* (4.11)*. Furthermore if* $\lambda < 0$*, then the solution* ψ *is unique.*

Proof. The conditions ($\mu_{p_Y} < \mu < \mu_{sc}$ or $0 < \mu$) imply that $1 < p := 2\mu\alpha + 1 < p_Y$, i.e. the problem (4.9) is superlinear but subcritical with respect to the Sobolev immersion theorem (see [29, Remark 5.5]). By recalling that $\psi = u^{\alpha}$, it is sufficient to prove that follows.

Let u_p be defined as in the case of ($1 < p < p_Y$) in Remark 4.8. If (λ, u) is a solution of (4.9), then multiplying (4.9) by u_p and integrating by parts there results

$$\beta\kappa_p \int_B u_p u dv_{g_B} = \lambda \int_B u_p u^p dv_{g_B}. \tag{4.15}$$

Thus $\mathrm{sign}(\lambda) = \mathrm{sign}(\kappa_p)$ since β, u_p and u are all positive.

Conversely, if λ is a real constant such that $\mathrm{sign}(\lambda) = \mathrm{sign}(\kappa_p) \neq 0$, then by *Remark 4.5*, (λ, u_λ) is a solution of (4.9), where $u_\lambda = t_\lambda u_p$ and $t_\lambda = \left(\dfrac{\lambda}{\beta\kappa_p}\right)^{\frac{1}{1-p}}$.

On the other side, if $\lambda = \kappa_p = 0$, then $(0, u_p)$ is a solution of (4.9).
Since $1 < p$, the uniqueness for $\lambda < 0$ is a consequence of Corollary 4.2. $\square$

Theorem 4.10. ($Cases$: $\mu = \mu_{p_Y}$) *If there exists* $\psi \in C^{\infty}_{>0}(B)$ *such that the scalar curvature of* $B \times_{[\psi^{\mu_{p_Y}},\psi]} F$ *is a constant* λ*, then* $sign(\lambda) = sign(\kappa_{p_Y})$*. Furthermore, if* $\lambda \in \mathbb{R}$ *verifying* $sign(\lambda) = sign(\kappa_{p_Y})$ *and* (4.13)*, then there exists* $\psi \in C^{\infty}_{>0}(B)$ *such that the scalar curvature of* $B \times_{[\psi^{\mu_{p_Y}},\psi]} F$ *is* λ*. Besides, if* $\lambda \in \mathbb{R}$ *is negative, then there exists at most one* $\psi \in C^{\infty}_{>0}(B)$ *such that the scalar curvature of* $B \times_{[\psi^{\mu_{p_Y}},\psi]} F$ *is* λ*.*

Proof. The proof is similar to that of Theorem 4.9, but follows from the application of the case of ($p = p_Y$) in Remark 4.8. Like above, the uniqueness of $\lambda < 0$ is a consequence of Corollary 4.2. $\square$

In the next proposition including the supercritical case, we will apply the following result (see also [73, p.99]).

Lemma 4.11. *Let (N_n, g_N) be a compact connected Riemannian manifold without boundary of dimension $n \geq 2$ and Δ_{g_N} be the corresponding Laplace-Beltrami operator. Consider the equation of the form*

$$\begin{gathered} -\Delta_{g_N} u = f(\cdot, u), \\ u \in C^{\infty}_{>0}(N) \end{gathered} \tag{4.16}$$

where $f \in C^{\infty}(N \times \mathbb{R}_{>0})$. If there exist a_0 and $a_1 \in \mathbb{R}_{>0}$ such that

$$\begin{gathered} u < a_0 \Rightarrow f(\cdot, u) > 0 \\ \text{and} \\ u > a_1 \Rightarrow f(\cdot, u) < 0, \end{gathered} \tag{4.17}$$

then (4.16) has a solution satisfying $a_0 \leq u \leq a_1$.

Proposition 4.12. *(Cases: $-\infty < \mu < \mu_{sc}$ or $0 < \mu$) If $\max S_B < 0$, then for all $\lambda < 0$ there exists $\psi \in C^{\infty}_{>0}(B)$ such that the scalar curvature of $B \times_{[\psi^{\mu}, \psi]} F$ is the constant λ. Furthermore, the solution ψ is unique.*

Proof. The conditions ($-\infty < \mu < \mu_{sc}$ or $0 < \mu$) imply that $1 < p := 2\mu\alpha + 1$.
On the other hand, since B is compact, by taking

$$f(.,u) = -S_B(\cdot)u + \lambda u^p = (-S_B + \lambda u^{p-1})u,$$

we obtain that $\lim_{u \longrightarrow 0^+} f(\cdot, u) = 0^+$ and $\lim_{u \longrightarrow +\infty} f(\cdot, u) = -\infty$. Thus (4.17) is verified.
Hence, the proposition is proved by applying Lemma 4.11 on (B_m, g_B). Notice that a_0 can take positive values and eventually gets close enough to 0^+ due to the condition of $\lim_{u \longrightarrow 0^+} f(\cdot, u)$, and consequently the corresponding solution results positive.
Again, since $\lambda < 0$ and $1 < p$ the uniqueness is a consequence of Corollary 4.2. ☐

Proof. (of Theorem 2.5) This is an immediate consequence of the above results. ☐

- *The case of a fiber with negative constant scalar curvature, i.e. $S_F < 0$.*

Here, the **(Pb-sc)** becomes equivalent to the study of the existence for the problem

$$\begin{gathered} -\beta \Delta_B u + S_B u = \lambda u^p - S_F u^q, \\ u \in C^{\infty}_{>0}(B), \end{gathered} \tag{4.18}$$

where λ is a real parameter (i.e., the searched constant scalar curvature), $\psi = u^{\alpha}$, $p = 2\mu\alpha + 1$ and $q = 2(\mu - 1)\alpha + 1$.

Remark 4.13. Let u be a solution of (4.18).

(i) If $\lambda_1 \leq 0$, then $\lambda < 0$. Indeed, multiplying the equation in (4.19) by u_1 and integrating by parts there results:

$$\lambda_1 \int_B u_1 u dv_{g_B} + S_F \int_B u_1 u^q dv_{g_B} = \lambda \int_B u_1 u^p dv_{g_B}, \tag{4.19}$$

where u_1 and u are positive.

(ii) If $\lambda = 0$, then $\lambda_1 > 0$.

(iii) If $\mu = 0$ (the warped product case), then $\lambda < \lambda_1$. These cases have been studied in [27, 24].

(iv) If $\mu = 1$ (the Yamabe problem for the usual product with conformal factor in $C^\infty_{>0}(B)$), there results $\operatorname{sign}(\lambda) = \operatorname{sign}(\lambda_1 + S_F)$.

An immediate consequence of Remark 4.13 is the following lemma.

Lemma 4.14. *Let B and F be given like in Theorem 2.3(i). Suppose further that B is a compact connected Riemannian manifold and F is a pseudo-Riemannian manifold of constant scalar curvature $S_F < 0$. If $\lambda \geq 0$ and $\lambda_1 \leq 0$ (for instance when $S_B \leq 0$ on B), then there is no $\psi \in C^\infty_{>0}(B)$ such that the scalar curvature of $B \times_{[\psi^\mu,\psi]} F$ is λ.*

Theorem 4.15. [29, Rows 6 and 8 in Table 4] *Under the hypothesis of Theorem 2.3(i), let B be a compact connected Riemannian manifold and F be a pseudo-Riemannian manifold of constant scalar curvature $S_F < 0$. Suppose that "$(m,k) \in D$ and $\mu \in (0,1)$" or "$(m,k) \in \mathcal{C}D$ and $\mu \in (0,1) \cap \mathcal{C}[\mu_-, \mu_+]$".*

(1) *If $\lambda_1 \leq 0$, then $\lambda \in \mathbb{R}$ is the scalar curvature of a $B \times_{[\psi^\mu,\psi]} F$ if and only if $\lambda < 0$.*

(2) *If $\lambda_1 > 0$, then there exists $\overline{\Lambda} \in \mathbb{R}_{>0}$ such that $\lambda \in \mathbb{R} \setminus \{\overline{\Lambda}\}$ is the scalar curvature of a $B \times_{[\psi^\mu,\psi]} F$ if and only if $\lambda < \overline{\Lambda}$.*

Furthermore if $\lambda \leq 0$, then there exists at most one $\psi \in C^\infty_{>0}(B)$ such that $B \times_{[\psi^\mu,\psi]} F$ has scalar curvature λ.

Proof. The proof of this theorem is the subject matter of §5. □

Once again we make use of Lemma 4.11 for the next theorem about the singular case and the following propositions.

Theorem 4.16. [29, Row 7 Table 4] *Under the hypothesis of Theorem 2.3(i), let B be a compact connected Riemannian manifold and F be a pseudo-Riemannian manifold of constant scalar curvature $S_F < 0$. Suppose that "$(m,k) \in \mathcal{C}D$ and $\mu \in (0,1) \cap (\mu_-, \mu_+)$", then for any $\lambda < 0$ there exists $\psi \in C^\infty_{>0}(B)$ such that the scalar curvature of $B \times_{[\psi^\mu,\psi]} F$ is λ. Furthermore the solution ψ is unique.*

Proof. First of all note that the conditions "$(m,k) \in \mathcal{C}D$ and $\mu \in (0,1) \cap (\mu_-, \mu_+)$" imply that $q < 0$ and $1 < p$, i.e. the problem (4.18) is superlinear in p but singular in q.

On the other hand, since B is compact, taking

$$f(.,u) = -S_B(\cdot)u + \lambda u^p - S_F u^q = [(-S_B(\cdot) + \lambda u^{p-1})u^{1-q} - S_F]u^q,$$

there result $\lim_{u \longrightarrow 0^+} f(\cdot,u) = +\infty$ and $\lim_{u \longrightarrow +\infty} f(\cdot,u) = -\infty$. Thus (4.17) is verified.

Thus by an application of Lemma 4.11 for (B_m, g_B), we conclude the proof for the existence part.

The uniqueness part just follows from Corollary 4.2. □

Remark 4.17. We observe that the arguments applied in the proof of Theorem 4.16 can be adjusted to the case of a compact connected Riemannian manifold B with $0 \leq q < 1 < p$, $\lambda < 0$ and $S_F < 0$, so that some of the situations included in Theorem 4.15. However, both argumentations are compatible but different.

Proof. (of Theorem 2.6) This is an immediate consequence of the above results. □

The approach in the next propositions is similar to Proposition 4.12 and Theorem 4.16.

Proposition 4.18. [29, Row 10 Table 4] *Let $1 < \mu < +\infty$. If $\max S_B < 0$, then for all $\lambda < 0$ there exists $\psi \in C^\infty_{>0}(B)$ such that the scalar curvature of $B \times_{[\psi^\mu, \psi]} F$ is the constant λ.*

Proof. The condition $1 < \mu < +\infty$ implies that $1 < q < p$.

On the other hand, since B is compact, taking

$$f(.,u) = -S_B(\cdot)u + \lambda u^p - S_F u^q = [-S_B(\cdot) + (\lambda u^{p-q} - S_F)u^{q-1}]u,$$

there result $\lim_{u \longrightarrow 0^+} f(\cdot,u) = 0^+$ and $\lim_{u \longrightarrow +\infty} f(\cdot,u) = -\infty$. Thus (4.17) is satisfied.

Thus an elementary application of Lemma 4.11 for (B_m, g_B) proves the proposition. □

Proposition 4.19. [29, Rows 2, 4 and 3 in Table 4] *Let either "$(m,k) \in D$ and $\mu \in (\mu_{sc}, 0)$" or "$(m,k) \in \mathcal{C}D$ and $\mu \in (\mu_{sc}, 0) \cap \mathcal{C}[\mu_-, \mu_+]$" or "$(m,k) \in \mathcal{C}D$ and $\mu \in (\mu_{sc}, 0) \cap (\mu_-, \mu_+)$". If $\min S_B > 0$, then for all $\lambda \leq 0$ there exists a smooth function $\psi \in C^\infty_{>0}(B)$ such that the scalar curvature of $B \times_{[\psi^\mu, \psi]} F$ is the constant λ.*

Proof. If either "$(m,k) \in D$ and $\mu \in (\mu_{sc}, 0)$" or "$(m,k) \in \mathcal{C}D$ and $\mu \in (\mu_{sc}, 0) \cap \mathcal{C}[\mu_-, \mu_+]$", then $0 < q < p < 1$.

On the other hand, since B is compact, taking

$$f(.,u) = -S_B(\cdot)u + \lambda u^p - S_F u^q = [-S_B(\cdot)u^{1-q} + \lambda u^{p-q} - S_F]u^q,$$

there result $\lim_{u\longrightarrow 0^+} f(\cdot,u) = 0^+$ and $\lim_{u\longrightarrow +\infty} f(\cdot,u) = -\infty$. Thus (4.17) is verified and again we can apply Lemma 4.11 for (B_m, g_B).

If "$(m,k) \in \mathcal{CD}$ and $\mu \in (\mu_{sc}, 0) \cap (\mu_-, \mu_+)$", then $q < 0 < p < 1$. Considering the limits as above, $\lim_{u\longrightarrow 0^+} f(\cdot,u) = +\infty$ and $\lim_{u\longrightarrow +\infty} f(\cdot,u) = -\infty$. So, an application of Lemma 4.11 concludes the proof. □

Remark 4.20. Notice that in Theorems 4.15 and 4.16 we do not assume hypothesis related to the sign of $S_B(\cdot)$, unlike in Propositions 4.12, 4.18 and 4.19.

Proposition 4.21. [29, Rows 5 and 9 in Table 4] *Let $(m,k) \in \mathcal{CD}$ be.*

(1) *If either "$\mu \in \left(-\dfrac{k}{m-1}, 0\right) \cap \{\mu_-, \mu_+\}$ and $\min S_B > 0$" or "$\mu \in (0,1) \cap \{\mu_-, \mu_+\}$", then for all $\lambda < 0$ there exists a smooth function $\psi \in C^\infty_{>0}(B)$ such that the scalar curvature of $B \times_{[\psi^\mu, \psi]} F$ is the constant λ. In the second case, ψ is also unique .*
(2) *If either "$\mu \in \left(-\dfrac{k}{m-1}, 0\right) \cap \{\mu_-, \mu_+\}$" or "$\mu \in (0,1) \cap \{\mu_-, \mu_+\}$" and furthermore $\lambda_1 > 0$, then there exists a smooth function $\psi \in C^\infty_{>0}(B)$ such that the scalar curvature of $B \times_{[\psi^\mu, \psi]} F$ is 0.*

Proof. In both cases $q = 0$, so by considering

$$f(.,u) = -S_B(\cdot)u + \lambda u^p - S_F,$$

the proof of (1) follows as in the latter propositions, while that of (2) is a consequence of the linear theory and the maximum principle. □

Remark 4.22. Finally, we observe a particular result about the cases studied in [27]. If $\mu = 0$, then $p = 1$ and $q = 1 - 2\alpha = \dfrac{k-3}{k+1}$. When the dimension of the fiber is $k = 2$, the exponent $q = -\dfrac{1}{3}$. So, writing the involved equation as

$$-\frac{8}{3}\Delta_B u = f(.,u) = -S_B(\cdot)u + \lambda u - S_F u^{-\frac{1}{3}}$$

and by applying Lemma 4.11 as above, we obtain that if $\lambda < \min S_B$, then there exists a smooth function $\psi \in C^\infty_{>0}(B)$ such that the scalar curvature of $B \times_\psi F$ is the constant λ. Furthermore, by Corollary 4.2 such ψ is unique (see [27, 24] and [25]).

5. Proof of the Theorem 4.15

The subject matter of this section is the proof of the Theorem 4.15, so we naturally assume its hypothesis.
Most of the time, we need to specify the dependence of λ of (4.18), we will do that by writing $(4.18)_\lambda$. Furthermore, we will denote the right hand side of $(4.18)_\lambda$ by $f_\lambda(t) := \lambda t^p - S_F t^q$.

The conditions either "$(m,k) \in D$ and $\mu \in (0,1)$" or "$(m,k) \in \mathcal{C}D$ and $\mu \in (0,1) \cap \mathcal{C}[\mu_-, \mu_+]$", imply that $0 < q < 1 < p$. But the type of nonlinearity in the right hand side of $(4.18)_\lambda$ changes with the sign λ, i.e. it is purely concave for $\lambda < 0$ and concave-convex for $\lambda > 0$.
The uniqueness for $\lambda \leq 0$ is again a consequence of Corollary 4.2.
In order to prove the existence of a solution for $(4.18)_\lambda$ with sign $\lambda \neq 0$, we adapt the approach of sub and upper solutions in [5].

Thus, the proof of Theorem 4.15 will be an immediate consequence of the results that follows.

Lemma 5.1. $(4.18)_0$ *has a solution if and only if* $\lambda_1 > 0$.

Proof. This situation is included in the results of the second case of Theorem 4.7 by replacing $-S_F$ with λ (see [24, Proposition 3.1]). □

Lemma 5.2. *Let us assume that* $\{\lambda : (4.18)_\lambda \text{ has a solution}\}$ *is non-empty and define*

$$\overline{\Lambda} = \sup\{\lambda : (4.18)_\lambda \text{ has a solution}\}. \tag{5.1}$$

- **(i)** *If* $\lambda_1 \leq 0$, *then* $\overline{\Lambda} \leq 0$.
- **(ii)** *If* $\lambda_1 > 0$, *then there exists* $\overline{\lambda} > 0$ *finite such that* $\overline{\Lambda} \leq \overline{\lambda}$.

Proof.

- **(i)** It is sufficient to observe *Remark 4.13 i.*
- **(ii)** Like in [5], let $\overline{\lambda} > 0$ such that

$$\lambda_1 t < \overline{\lambda} t^p - S_F t^q, \forall t \in \mathbb{R}, t > 0. \tag{5.2}$$

Thus, if (λ, u) is a solution of $(4.18)_\lambda$, then

$$\lambda \int_B u_1 u^p - S_F \int_B u_1 u^q = \int_B \lambda_1 u_1 u < \overline{\lambda} \int_B u_1 u^p - S_F \int_B u_1 u^q,$$

so $\lambda < \overline{\lambda}$.

□

Lemma 5.3. *Let*

$$\overline{\Lambda} = \sup\{\lambda : (4.18)_\lambda \text{ has a solution}\}. \tag{5.3}$$

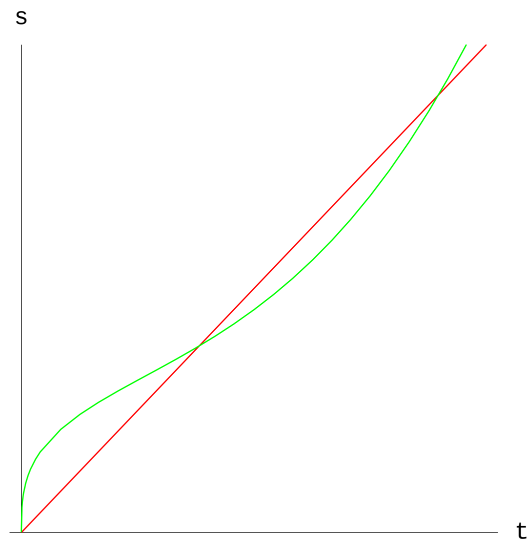

FIGURE 1. The nonlinearity f_λ in Lemma 5.3, i.e. $0 < q < 1 < p, S_F < 0, \lambda_1 > 0, \lambda > 0$.

(i) *Let $E \in \mathbb{R}_{>0}$. There exist $0 < \lambda_0 = \lambda_0(E)$ and $0 < M = M(E, \lambda_0)$ such that $\forall \lambda : 0 < \lambda \leq \lambda_0$, so we have*

$$0 < E\frac{f_\lambda(EM)}{EM} < 1. \tag{5.4}$$

(ii) *If $\lambda_1 > 0$, then $\{\lambda > 0 : (4.18)_\lambda$ has a solution$\} \neq \emptyset$. As a consequence of that, $\overline{\Lambda}$ is finite.*

(iii) *If $\lambda_1 > 0$, then for all $0 < \lambda < \overline{\Lambda}$ there exists a solution of the problem $(4.18)_\lambda$.*

Proof.

(i) For any $0 < \lambda < \lambda_0$

$$\begin{aligned} 0 < g_\lambda(r) := E\frac{f_\lambda(Er)}{Er} &= Er^{q-1}(\lambda E^{p-1}r^{p-q} - S_F E^{q-1}) \\ &< Er^{q-1}(\lambda_0 E^{p-1}r^{p-q} - S_F E^{q-1}). \end{aligned}$$

It is easy to see that

$$r_0 = \left(\frac{S_F}{\lambda_0}\frac{q-1}{p-1}\right)^{\frac{1}{p-q}} \frac{1}{E}$$

is a minimum point for g_{λ_0} and

$$g_{\lambda_0}(r_0) = E\left(\frac{S_F}{\lambda_0}\frac{q-1}{p-1}\right)^{\frac{q-1}{p-q}} S_F\left[\frac{q-1}{p-1}-1\right] \to 0^+, \text{ as } \lambda_0 \to 0^+.$$

Hence there exist $0 < \lambda_0 = \lambda_0(E)$ and $0 < M = M(E, \lambda_0)$ such that (5.4) is verified.

(ii) Since $\lambda_1 > 0$, by the maximum principle, there exists a solution $e \in C^\infty_{>0}(B)$ of

$$L_B(e) = -\beta\Delta_B e + S_B e = 1. \tag{5.5}$$

Then, applying item (i) above with $E = \|e\|_\infty$ there exists $0 < \lambda_0 = \lambda_0(\|e\|_\infty)$ and $0 < M = M(\|e\|_\infty, \lambda_0)$ such that $\forall\lambda$ with $0 < \lambda \leq \lambda_0$ we have that

$$L_B(Me) = M \geq f_\lambda(Me), \tag{5.6}$$

hence Me is a supersolution of $(4.18)_\lambda$.

On the other hand, since $\check{u}_1 := \inf u_1 > 0$, for all $\lambda > 0$

$$\epsilon^{-1} f_\lambda(\epsilon\check{u}_1) = \epsilon^{q-1}[\lambda\epsilon^{p-q}\check{u}_1^p - S_F\check{u}_1^q] \to +\infty, \text{ as } \epsilon \to 0^+. \tag{5.7}$$

Furthermore, note that f_λ is nondecreasing when $\lambda > 0$. Hence for any $0 < \lambda$ there exists a small enough $0 < \epsilon$ verifying

$$L_B(\epsilon u_1) = \epsilon\lambda_1 u_1 \leq \epsilon\lambda_1\|u_1\|_\infty \leq f_\lambda(\epsilon\check{u}_1) \leq f_\lambda(\epsilon u_1), \tag{5.8}$$

thus ϵu_1 is a subsolution of $(4.18)_\lambda$.

Then for any $0 < \lambda < \lambda_0$, (taking eventually $0 < \epsilon$ smaller if necessary), we have that the above constructed couple sub super solution satisfies

$$\epsilon u_1 < Me. \tag{5.9}$$

Now, by applying the monotone iteration scheme, we have that $\{\lambda > 0 : (4.18)_\lambda \text{ has a solution}\} \neq \emptyset$. Furthermore by Lemma 5.2 (ii) there results $\overline{\Lambda}$ is finite.

(iii) The proof of this item is completely analogous to *Lemma 3.2* in [5]. We will rewrite this to be self contained.

Given $\lambda < \overline{\Lambda}$, let u_ν be a solution of $(4.18)_\nu$ with $\lambda < \nu < \overline{\Lambda}$. Then u_ν is a supersolution of $(4.18)_\lambda$ and for small enough $0 < \epsilon$, the subsolution ϵu_1 of $(4.18)_\lambda$ verifies $\epsilon u_1 < u_\nu$, then as above $(4.18)_\lambda$ has a solution.

□

Lemma 5.4. *For any $\lambda < 0$, there exists $\gamma_\lambda > 0$ such that $\|u\|_\infty \leq \gamma_\lambda$ for any solution u of $(4.18)_\lambda$. Furthermore if S_B is nonnegative, then positive zero of f_λ can be choose as γ_λ.*

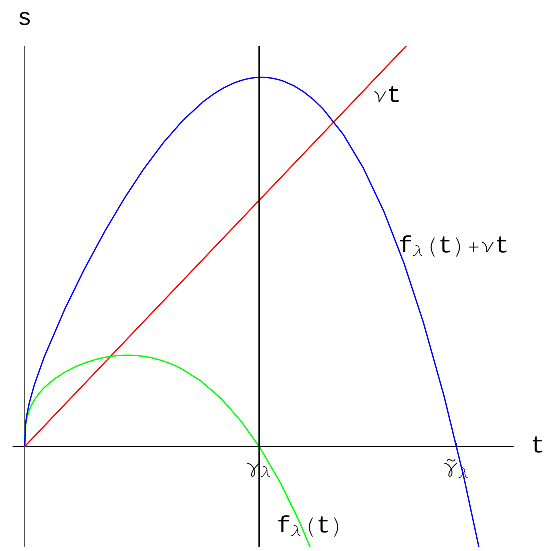

FIGURE 2. The nonlinearity in Lemma 5.5 , i.e. $0 < q < 1 < p$, $S_F < 0$, $\lambda_1 > 0$, $\lambda < 0$.

Proof. Define $\check{S}_B := \min S_B$ (recall that B is compact). There are two different situations, namely.

- $\underline{0 \leq \check{S}_B}$: since there exists $x_1 \in B$ such that $u(x_1) = \|u\|_\infty$ and $0 \leq -\beta\Delta_B u(x_1) = -S_B(x_1)\|u\|_\infty + \lambda\|u\|_\infty^p - S_F\|u\|_\infty^q$, there results $\|u\|_\infty \leq \gamma_\lambda$, where γ_λ is the strictly positive zero of f_λ.
- $\underline{\check{S}_B < 0}$: we consider $\widetilde{f_\lambda}(t) := \lambda t^p - S_F t^q - \check{S}_B t$. Now, our problem $\underline{(4.18)_\lambda}$ is equivalent to

$$-\beta\Delta_B u + (S_B - \check{S}_B)u = \widetilde{f_\lambda}(u),$$
$$u \in C^\infty_{>0}(B).$$

  But here the potential of $(S_B - \check{S}_B)$ is non negative and the function $\widetilde{f_\lambda}$ has the same behavior of f_λ with a positive zero $\widetilde{\gamma_\lambda}$ on the right side of the positive zero γ_λ of f_λ. Thus, repeating the argument for the case of $\check{S}_B \geq 0$, we proved $\|u\|_\infty \leq \tilde{\gamma}_\lambda$.

□

Lemma 5.5. *Let $\lambda_1 > 0$. Then for all $\lambda < 0$ there exists a solution of* $(4.18)_\lambda$.

Proof. We will apply again the monotone iteration scheme. Define $\check{S}_B := \min S_B$ (note that B is compact).

- $\underline{0 \leq \check{S}_B}$: Clearly, the strictly positive zero γ_λ of f_λ is a supersolution of

$$-\beta\Delta_B u + (S_B + \nu)u = f_\lambda(u) + \nu u, \tag{5.10}$$

for all $\nu \in \mathbb{R}$.

On the other hand, for $0 < \epsilon = \epsilon(\lambda)$ small enough,

$$L_B(\epsilon u_1) = \epsilon\lambda_1 u_1 \leq f_\lambda(\epsilon u_1). \tag{5.11}$$

Then ϵu_1 is a subsolution of (5.10) for all $\nu \in \mathbb{R}$.

By taking ε possibly smaller, we also have

$$0 < \epsilon u_1 < \gamma_\lambda. \tag{5.12}$$

We note that for large enough values of $\nu \in \mathbb{R}_{>0}$, the nonlinearity on the right hand side of (5.10), namely $f_\lambda(t) + \nu t$, is an increasing function on $[0, \gamma_\lambda]$.

Thus applying the monotone iteration scheme we obtain a strictly positive solution of (5.10), and hence a solution of $(4.18)_\lambda$ (see [3], [4], [54]).

- $\underline{\check{S}_B < 0}$: In this case, like in Lemma 5.4 we consider $\widetilde{f_\lambda}(t) := \lambda t^p - S_F t^q - \check{S}_B t$. Then, the problem $(4.18)_\lambda$ is equivalent to

$$\begin{aligned} -\beta\Delta_B u + (S_B - \check{S}_B)u &= \widetilde{f_\lambda}(u), \\ u &\in C^\infty_{>0}(B), \end{aligned} \tag{5.13}$$

where the potential is nonnegative and the function $\widetilde{f_\lambda}$ has a similar behavior to f_λ with a positive zero $\widetilde{\gamma_\lambda}$ on the right side of the positive zero γ_λ of f_λ.

Here, it is clear that $\tilde{\gamma}_\lambda$ is a positive supersolution of

$$-\beta\Delta_B u + (S_B - \check{S}_B + \nu)u = \widetilde{f_\lambda}(u) + \nu u, \tag{5.14}$$

for all $\nu \in \mathbb{R}$. Hence, we complete the proof similarly to the case of $\check{S}_B \geq 0$.

□

Lemma 5.6. *Let $\lambda_1 \leq 0$, $\lambda < 0$, $\check{S}_B := \min S_B$ and also let γ_λ be a positive zero of f_λ and $\tilde{\gamma}_\lambda$ be a positive zero of $\tilde{f}_\lambda := f_\lambda - \check{S}_B id_{\mathbb{R}_{\geq 0}}$. Then there exists a solution u of $(4.18)_\lambda$. Furthermore any solution of $(4.18)_\lambda$ satisfies $\gamma_\lambda \leq \|u\|_\infty \leq \tilde{\gamma}_\lambda$.*

Proof. First of all we observe that if $S_B \equiv 0$ (so $\lambda_1 = 0$), then $u \equiv \gamma_\lambda$ is the searched solution of $(4.18)_\lambda$.

Now, we assume that $S_B \not\equiv 0$. Since $\lambda_1 \leq 0$, there results $\check{S}_B < 0$. In this case, one can notice that $0 < \gamma_\lambda < \tilde{\gamma}_\lambda$.

On the other hand, the problem $(4.18)_\lambda$ is equivalent to

$$-\beta\Delta_B u + (S_B - \check{S}_B)u = \widetilde{f}_\lambda(u), \quad u \in C^\infty_{>0}(B). \tag{5.15}$$

By the second part of the proof of Lemma 5.4, if u is a solution of $(4.18)_\lambda$ (or equivalently (5.15)), then $\|u\|_\infty \leq \tilde{\gamma}_\lambda$. Besides, since

$$\int_B u_1(f_\lambda \circ u) = \lambda_1 \int_B u_1 u,$$

$u, u_1 > 0$ and $\lambda_1 \leq 0$ results $\gamma_\lambda \leq \|u\|_\infty$.
From this point on, the proof of the existence of solutions for (5.15) follows the lines of the second part of Lemma 5.5. □

6. Conclusions and future directions

Now, we would like to summarize the content of the paper and to propose our future plans on this topic.

We remark to the reader that several computations and proofs, along with other complementary results mentioned in this article and references can be obtained in [29]. We have chosen this procedure to avoid the involved long computations.

In brief, we introduced and studied curvature properties of a particular family of warped products of two pseudo-Riemannian manifolds which we called as a *base conformal warped product*. Roughly speaking the metric of such a product is a mixture of a conformal metric on the base and a warped metric. We concentrated our attention on a special subclass of this structure, where there is a specific relation between the *conformal factor* c and the *warping function* w, namely $c = w^\mu$ with μ a real parameter.
As we mentioned in §1 and the first part of §2, these kinds of metrics and considerations about their curvatures are very frequent in different physical areas, for instance theory of general relativity, extra-dimension theories (Kaluza-Klein, Randall-Sundrum), string and super-gravity theories; also in global analysis for example in the study of the spectrum of Laplace-Beltrami operators on p-forms, etc.

More precisely, in Theorems 3.1 and 3.2, we obtained the classical relations among the different involved Ricci tensors (respectively, scalar curvatures) for metrics of the form $c^2 g_B \oplus w^2 g_F$. Then the study of particular families of either scalar or tensorial nonlinear partial differential operators on pseudo-Riemannian manifolds (see Lemmas 3.3 and 3.7) allowed us to find reduced expressions of the Ricci tensor and scalar curvature for metrics as above with $c = w^\mu$, where μ a real parameter (see Theorems 2.2 and 2.3). The operated reductions can be considered as generalizations of those used by Yamabe in [79] in order to obtain the transformation law of the scalar curvature under

a conformal change in the metric and those used in [27] with the aim to obtain a suitable relation among the involved scalar curvatures in a singly warped product (see also [52] for other particular application and our study on multiply warped products in [28]).

In §4 and 5, under the hypothesis that (B, g_B) be a "compact" and connected Riemannian manifold of dimension $m \geq 3$ and (F, g_F) be a pseudo - Riemannian manifold of dimension $k \geq 0$ with constant scalar curvature S_F, we dealt with the problem **(Pb-sc)**. This question leads us to analyze the existence and uniqueness of solutions for nonlinear elliptic partial differential equations with several kinds of nonlinearities. The type of nonlinearity changes with the value of the real parameter μ and the sign of S_F. In this article, we concentrated our attention to the cases of constant scalar curvature $S_F \leq 0$ and accordingly the central results are Theorems 2.5 and 2.6. Although our results are partial so that there are more cases to study in forthcoming works, we obtained also other complementary results under more restricted hypothesis about the sign of the scalar curvature of the base.

Throughout our study, we meet several types of partial differential equations. Among them, most important ones are those with concave-convex nonlinearities and the one so called Lichnerowicz-York equation. About the former, we deal with the existence of solutions and leave the question of multiplicity of solutions to a forthcoming study.

We observe that the previous problems as well as the study of the Einstein equation on *base conformal warped products*, (ψ, μ)-bcwp*'s* and their *generalizations to multi-fiber cases*, give rise to a reach family of interesting problems in differential geometry and physics (see for instance, the several recent works of R. Argurio, J. P. Gauntlett, M. O. Katanaev, H. Kodama, J. Maldacena, H. -J. Schmidt, A. Strominger, K. Uzawa, P. S. Wesson among many others) and in nonlinear analysis (see the different works of A. Ambrosetti, T. Aubin, I. Choquet-Bruat, J. Escobar, E. Hebey, J. Isenberg, A. Malchiodi, D. Pollack, R. Schoen, S. -T. Yau among others).

Appendix A.

Let us assume the hypothesis of Theorem 2.3 (i), the dimensions of the base $m \geq 2$ and of the fiber $k \geq 1$. In order to describe the classification of the type of nonlinearities involved in (2.11), we will introduce some notation (for a complete study of these nonlinearities see [29, Section 5]). The example in Figure 1 will help the reader to clarify the notation.

Note that the denominator in (2.12) is

$$\eta := (m-1)(m-2)\mu^2 + 2(m-2)k\mu + (k+1)k \tag{A.1}$$

and verifies $\eta > 0$ for all $\mu \in \mathbb{R}$. Thus α in (2.12) is positive if and only if $\mu > -\dfrac{k}{m-1}$ and by the hypothesis $\mu \neq -\dfrac{k}{m-1}$ in Theorem 2.3 (i), results $\alpha \neq 0$.

We now introduce the following notation:

$$\begin{aligned} p = p(m,k,\mu) = \quad & 2\mu\alpha + 1 \text{ and} \\ q = q(m,k,\mu) = \quad & 2(\mu-1)\alpha + 1 = p - 2\alpha, \end{aligned} \tag{A.2}$$

where α is defined by (2.12).

Thus, for all m, k, μ given as above, p is positive. Indeed, by (A.1), $p > 0$ if and only if $\varpi > 0$, where

$$\begin{aligned} \varpi \quad &:= \varpi(m,k,\mu) \\ &:= 4\mu[k + (m-1)\mu] + (m-1)(m-2)\mu^2 + 2(m-2)k\mu + (k+1)k \\ &= (m-1)(m+2)\mu^2 + 2mk\mu + (k+1)k. \end{aligned}$$

But *discr* $(\varpi) \leq -4km^2 \leq -16$ and $m > 1$, so $\varpi > 0$.

Unlike p, q changes sign depending on m and k. Furthermore, it is important to determine the position of p and q with respect to 1 as a function of m and k. In order to do that, we define

$$D := \{(m,k) \in \mathbb{N}_{\geq 2} \times \mathbb{N}_{\geq 1} : \mathit{discr}\ (\varrho(m,k,\cdot)) < 0\}, \tag{A.3}$$

where $\mathbb{N}_{\geq l} := \{j \in \mathbb{N} : j \geq l\}$ and

$$\begin{aligned} \varrho \quad &:= \varrho(m,k,\mu) \\ &:= 4(\mu-1)[k + (m-1)\mu] + (m-1)(m-2)\mu^2 + 2(m-2)k\mu + (k+1)k \\ &= (m-1)(m+2)\mu^2 + 2(mk - 2(m-1))\mu + (k-3)k. \end{aligned}$$

Note that by (A.1), $q > 0$ if and only if $\varrho > 0$. Furthermore $q = 0$ if and only if $\varrho = 0$. But here *discr* $(\varrho(m,k,\cdot))$ changes its sign as a function of m and k.

We adopt here the notation in [29, Table 4] below, namely $\mathcal{C}D = (\mathbb{N}_{\geq 2} \times \mathbb{N}_{\geq 1}) \setminus D$ if $D \subseteq \mathbb{N}_{\geq 2} \times \mathbb{N}_{\geq 1}$ and $\mathcal{C}I = \mathbb{R} \setminus I$ if $I \subseteq \mathbb{R}$. Thus, if $(m,k) \in \mathcal{C}D$, let μ_- and μ_+ two roots (eventually one, see [29, Remark 5.3]) of q, $\mu_- \leq \mu_+$. Besides, if *discr* $(\varrho(m,k,\cdot)) > 0$, then $\mu_- < 0$; whereas μ_+ can take any sign.

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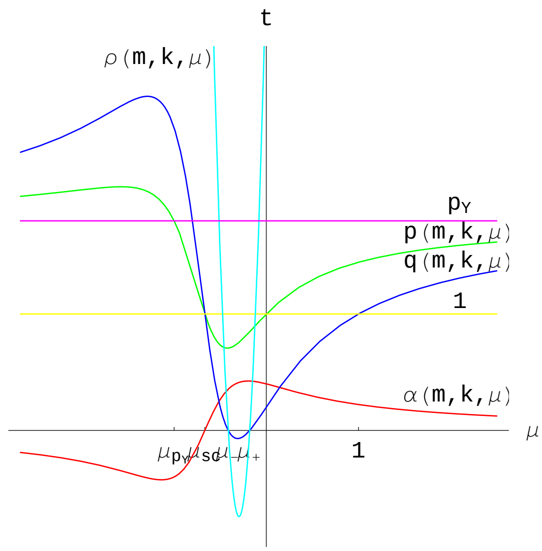

FIGURE 3. Example: $(m, k) = (7, 4) \in \mathcal{C}D$

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(F. Dobarro) Dipartimento di Matematica e Informatica, Università degli Studi di Trieste, Via Valerio 12/b, I-34127 Trieste, Italy

E-mail address: `dobarro@dmi.units.it`

(B. Ünal) Department of Mathematics, Bilkent University, Bilkent, 06800 Ankara, Turkey

E-mail address: `bulentunal@mail.com`